The Advanced Econometrics of Continuous Risk-Free Interest Rate Modelling and Implementation for Life Insurance Valuation

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Abstract

Most analytically driven interest rate derivations initiated in life contingencies are deemed fair to life insurers in meeting solvency requirements which arguably may not be satisfactorily fair to the insured. The arguments presented in this paper are written from the practical underwriting perspectives and are aimed at circumventing the comparatively intractable process of interest rate computations. The objectives are to obtain the hedge ratio each term of which is based on the uniform distribution of death assumption, use this ratio to derive power series in terms of Bernoulli numbers, estimate the risk-free interest rate intensities from the power series and compare the exact result with the estimated results. Given a tolerance limit of 0.4%, computational evidence shows that the absolute deviation of the exact from the estimated interest rate is less than 0.4%, it may be fairer to the insured in the performance of actuarial valuation involving present values computations.

Keywords: *Life Annuities, Force of Interest, Power Series, Bernoulli Numbers, underwriting, Valuations*

JEL Classification: C1, C4, C5

Introduction

Interest rate is widely applied to perform valuation in order to price life insurance products, especially in computing interest rate driven life annuities, life insurance to ensure solvency of life insurers through sufficient reserves. In life insurance valuation, interest rate is assumed constant, particularly in the default free deterministic market, where premium setting is characterized by the principle of equivalence. Past works which concentrated on Taylor's series to estimate interest rate intensity could not successively arrive at a closed form formula and could not investigate whether the results were reasonably accurate for the valuation of life insurance underwriting purposes.

Actuaries and other insurance practitioners pick predetermined interest rate a priori to soothe life insurance valuation. This constitutes a problem in the quest for the development of functional forms of the deterministic interest rate function. In the field of computational insurance, efficient estimation technique is required to price complicated policies and calibrate actuarial models. These two areas rely on robust numerical techniques and rapidly efficient modelling and computations of interest rate. Interest rate scenario represents a very critical assumption of life insurance cash flow testing results. The deterministic market is default free in classical life contingency domain and consequently, the valuation of interest rate is constant. This study is timely for the Nigerian annuity market as the federal Government had recently enacted laws to regulate the Nigerian annuity market. The default free market interest rate scenarios are employed in life insurance since interest rates are fixed during actuarial valuation. Although the deterministic interest rate scenario may not moderately represent adverse conditions, extant actuarial literature contains no extensive formal exploration of this area.

Literature review

As a result of the recent instability in the insurance markets, the insurance companies try to optimize and accelerate underwriting operational processes to remain competitive. The attainment of this goal requires a deep knowledge of interest rate modelling. Following Udoye et al. (2021), a great deal of effort has been deployed to obtain new algorithms and new financial models or even modify the existing models so as to defuse their complexity while still maintaining the same level of computational accuracy. An emerging computationally intensive problem that life insurers encounter is the modelling of interest rate applicable in life insurance valuation.

There is a gradual paradigm shift of insurance sector from the direct regulatory control system to a more liberalized environment, hence requiring new financial risk management. The regulatory authorities need to employ improved and advanced techniques for regulating product development through robust interest rate mechanism. Insurance institutions are the main investors and their credibility has marked implications on the financial stability; as such the main reference point of an insurer is its financial health or solvency. The obligation of an insurer is the operational underwriting processes and their expected expenses which are often computed using actuarial techniques for the protection of the policy holders and to guarantee of the stability of the insurance market.

Present value problems driven by interest rate computations represent a crucial analytical tool in life insurance mathematics. Following observations in Panjer and Bellhouse (1980); Belhouse and Panjer (1981) and Jothi (2009) and Kellison (2009), the two major applications of this area concern issues relating to estimation and modelling. The latter has a longer pedigree as studied in this paper and becomes an extensive area of life contingencies (Anggraeni et al. 2023). Following observations in Kellison (2009), McCutcheon and Scott (1986), the deepest form of this estimation problems remains a subject of classical life insurance methodologies. In practice, crucial transactions where interest rate estimation is applicable may possibly become analytically intractable in the quest for a closed form solution (Jothi, 2009). Furthermore, in the computation of the present value $e^{-\delta}$ of benefits, the value of δ may be very high and may not be consistent with the insured's benefits. Consequently, in this instance estimation methods for δ becomes an inevitable mathematical tool.

From the observation raised, the application of the Euler-Maclaurin series appears to be an interesting estimation technique in interest rate modelling. To embed this in the process of insurance valuation, the nominal rate of interest approach can be perceived as an approximation of the force of interest by the truncation of the Euler Maclaurin series as appropriate. Interest rate estimations are important based on two major reasons. The first reason evolves from the gap between numerical estimations and analytical investigation. While numerical analysis sheds light on definite interest rate scenarios analytical techniques considers important properties and general behaviour.

The latter includes asymptotic behaviour of solution as the interest rates become large or infinitesimally small. The second reason is the challenge of implementing the approximation schemes. Therefore, this paper contributes in both ways as the results evince better grasp of such estimation techniques. Consequently, the effect of the Euler Maclaurin series on the behaviour of nominal rate of interest in the short and long run is investigated. This method is important because of the possibility of generating a closed form solution that serves as a reference point in a more complex interest rate scenario. According to Anggraeni *et al.* (2023), actuarial valuation computation can be perceived as a consolidation of future cash flows projects anticipated cash flows on a yearly basis. The cash flows are analyzed on the basis that survival and mortality probabilities and economic assumptions of salary inflation and pension increases. The premium, investment incomes and benefit outgone in each year are then discounted using the valuation rate of interest effective the date of valuation to obtain the present values of premium income and benefit outgone.

In Anggraeni *et al*. (2023), the rationale behind performing actuarial valuation of a scheme is to investigate how well the assets cover the liabilities. This procedure is important since a pension plan cannot continue indefinitely. If a scheme is discounted, its assets at that material time is crucial. Consequently, if the plan asset is not sufficient, members will not earn the expected benefits since the examination of present values may not necessarily elicit the long-run benefit of the cash flows being obtained.

It is imperative to discount the future payments at the valuation rate of interest to obtain an appropriate present value for actuarial computations. In Jothi (2009) and Kellison (2009), a discounted cash flow process is applied for future benefit payments and premiums for future investment proceeds. The advantage of this technique is that assets are valued in a way that those are consistent with the actuarial liabilities. In actuarial life problems, discounting is based on the theory of interest rate intensities; this gives vent to the need for its thorough examination. The objectives are (i) to obtain the hedge ratio $\bar{A}_x / A_x$ each term of which is based on the uniform distribution of death assumption (ii) to use this ratio to derive power series in terms of Bernoulli numbers (iii) to estimate the risk-free interest rate intensities from the power series and (iv) to compare the exact result with the estimated results.

The Present value problem

Let d define the discount rate, then the present value PV of a sum C due in s years at the nominal rate of discount d convertible n times per year is defined as

$$PV = C\left(1-\frac{d}{n}\right)^{ns} \tag{1a}$$

$$\lim_{n\to\infty} PV = \lim_{n\to\infty}\left(C\left(1-\frac{d}{n}\right)^{ns}\right) \tag{1b}$$

$$\Rightarrow PV = C\lim_{n\to\infty}\left(\left(1-\frac{d}{n}\right)^{ns}\right) \tag{1c}$$

$$\Rightarrow PV = C\left(\lim_{n\to\infty}\left(1-\frac{d}{n}\right)^{-(ns)\left(\frac{-d}{d}\right)}\right) \tag{1d}$$

$$\Rightarrow PV = C\left(\lim_{n\to\infty}\left(1-\frac{d}{n}\right)^{-\left(\frac{n}{d}\right)(-ds)}\right) \tag{1e}$$

$$\Rightarrow PV = C\left(\left[\lim_{n\to\infty}\left(1+\left(-\frac{d}{n}\right)\right)^{\left(\frac{1}{\frac{-d}{n}}\right)}\right]^{(-ds)}\right) \tag{1f}$$

Letting $-nx = d$

$$\lim_{n\to\infty} PV = C\left(\left[\lim_{n\to\infty}(1+x)^{\frac{1}{x}}\right]^{(-ds)}\right) \tag{1g}$$

The present value of an obligation of C due in s years' time at d nominal rate of discount convertible continuously is obtained as $PV = Ce^{-ds}$

Justifications for modelling the force of interest

Modelling the force of interest is a key component in the valuation of life insurance products, as it is a crucial tool for calculating the time value of money over the policy's lifetime. The force of interest is a continuous rate of interest that is used to model how the value of money changes over time in a life insurance contract. Below are several key reasons why it is important to model the force of interest in life insurance valuation:

The force of interest allows actuaries to discount future cash flows such as premiums, claims, or benefits in a continuous manner. This is important because life insurance contracts involve long time horizons, with premiums and benefits occurring over many years or decades. A continuous discounting model ensures that the time value of money is reflected accurately.

The force of interest is based on continuous compounding, which aligns with modern financial theory. Continuous compounding better approximates real-world interest rate movements and provides a more precise measure for pricing and valuing long-term financial contracts, such as life insurance.

Life insurance policies involve long-term commitments. The force of interest quantifies how the value of money changes over time due to interest accumulation. By applying the force of interest, actuaries can assess the time value of future cash flows, accounting for inflation, opportunity cost, and investment returns.

Using the force of interest allows for more flexibility in modeling interest rate environments and investment strategies. It provides a continuous and smooth representation of interest rate changes over time, as opposed to discrete models (such as annual compounding) that may introduce artificial discontinuities in the valuation process.

The force of interest can be combined with stochastic interest rate models, which allow actuaries to assess the impact of interest rate fluctuations on the valuation of life insurance products. This is particularly useful when testing different economic scenarios, such as changes in interest rates, inflation, or market volatility.

In life insurance, the force of interest is integral to both pricing and reserving. For pricing, it helps determining the appropriate premiums for policyholders given the time value of benefits and expenses. For reserving, it allows insurers to calculate the present value of future policyholder obligations, ensuring that they hold sufficient reserves to meet future claims.

The continuous nature of the force of interest better reflects the realities of financial markets where interest rates tend to change continuously rather than in discrete time periods. It provides a closer approximation of actual investment returns, which tend to be realized continuously in practice (e.g reinvestment of cash flows, dividends, etc.).

Life insurance policies typically have long durations, and the force of interest helps insurers plan for future obligations over those periods. It assists in managing cash flows, ensuring solvency, and understanding how liabilities evolve over time under different interest rate assumptions.

Many modern life insurance products, such as universal life or variable annuities, have complex benefit structures with embedded options (e.g., investment-linked benefits, death benefits, or withdrawal options). The force of interest is useful in valuing these products, particularly when the benefits or cash flows are contingent on interest rates or other market factors.

Many regulatory frameworks and accounting standards require life insurers to use sophisticated models for the valuation of liabilities. For instance, under Solvency II (in Europe) or IFRS 17 (international accounting standards), continuous discounting based on the force of interest might be required to assess the present value of future insurance liabilities.

Bernoulli numbers

The Bernoulli numbers B_m are usually expressed as coefficients of power series. Following Coen (1996); Apostol (2008); Howard (1995); Franjic, Pecaric (2005); Marco et al. (2018),

$$\frac{y}{e^y-1}=\sum_{m=0}^{\infty} B_m \frac{y^m}{m!}; \quad y \neq 0 \tag{2a}$$

The coefficients in $\sum_{m=0}^{\infty} B_m \frac{y^m}{m!}$ are difficult to obtain, consequently, the reciprocal of the term $\frac{y}{e^y-1}$ can be expanded using the Taylor's series expansion as follows:

$$\frac{e^y-1}{y}=\frac{1}{y}\sum_{k=1}^{\infty}\frac{y^k}{k!} \tag{2b}$$

$$\frac{e^y-1}{y}=\sum_{k=0}^{\infty}\frac{y^k}{(k+1)!} \tag{3}$$

$$\Rightarrow \frac{e^y - 1}{y} = 1 + \frac{y}{2} + \frac{y^2}{6} + \frac{y^3}{24} + \dots \tag{4}$$

But

$$\left(\frac{e^y - 1}{y}\right)\left(\frac{y}{e^y - 1}\right) = 1 \tag{5}$$

Observe that from (5)

$$\left(\frac{e^y - 1}{y}\right)\left(\frac{y}{e^y - 1}\right) = \sum_{m,k=0}^{\infty} B_m \frac{y^{m+k}}{m!(k+1)!} \tag{6}$$

Equations (2a) and (5) imply

$$\left(\frac{e^y - 1}{y}\right)\left(\frac{y}{e^y - 1}\right) = \left(1 + \frac{y}{2} + \frac{y^2}{6} + \frac{y^3}{24} + \dots\right)\left(B_0 + \frac{B_1}{1!} y + \frac{B_2}{2!} y^2 + \frac{B_3}{3!} y^3 \dots\right) = 1 \tag{7}$$

$$\left(\frac{e^y - 1}{y}\right)\left(\frac{y}{e^y - 1}\right) = B_0 + \left(\frac{B_0}{2!0!} + \frac{B_1}{1!1!}\right) y + \left(\frac{B_0}{3!0!} + \frac{B_1}{2!1!} + \frac{B_2}{1!2!}\right) y^2 + \dots = 1 \tag{8}$$

Comparison of coefficients (setting of B_0 to 1 and coefficients of other powers of y to 0) yielded

$B_0 = 1$; $B_1 = -\frac{1}{2} B_0 = -\frac{1}{2}$; $B_2 = -\frac{1}{3} B_0 - B_1 = \frac{1}{6}$; $B_1 = -\frac{1}{2}$; $B_4 = -\frac{1}{30}$;
$B_6 = -\frac{1}{42}$; $B_8 = -\frac{1}{30}$;

$B_{10} = -\frac{5}{66}$

$$B_{2m} = 2(-1)^{m-1} \frac{(2m)!}{(2\pi)^{2m}} \times \varsigma(2m) = 2(-1)^{m-1} \frac{(2m)!}{(2\pi)^{2m}} \times \left(1 + O\left(2^{-2m}\right)\right) \tag{9}$$

$B_{2m+1} = 0; \quad m \geq 1$

$$\varsigma(2m) = \sum_{r=1}^{\infty} r^{-2m} = \frac{|B_{2m}|(2\pi)^{2m}}{2(2m)!} \tag{9a}$$

Thus, based on the observations in Lehmer (1988); Tuenter (2001); Si (2019), Guo and Liu (2020), the following condition in (9b) is valid

$$\sum_{m=0}^{k} \frac{B_m}{m!\left(k+1-m\right)!} = 0; \quad k \geq 1 \tag{9b}$$

$$\frac{y}{e^y - 1} = 1 - \frac{y}{2} + \frac{y^2}{12} - \frac{y^4}{720} + \frac{y^6}{30240} - \frac{y^8}{1209600} + \frac{y^{10}}{47900160} \tag{10}$$

Material and methods

Modelling the default risk interest rate intensity through the evolution of Bernoulli series

In practice the accumulation function $A(s)$ is usually not given and in particular, nominal interest rate is constant: $\delta(s) = \delta$ in life insurance underwriting because in the deterministic market, interest rate is default free. Consequently, it is necessary to construct a reasonable model for the force of interest δ based on Bernoulli Series formula. Accordingly, the following is a theorem due to the authors of this paper.

Theorem

(i) If $m \in \mathbf{Z}^+$ for $\delta > 0$ then the following relationship holds

$$\frac{m}{a_{\overline{m}|}} = \frac{m}{\left\{\dfrac{1-e^{-m\delta}}{\delta}\right\}} \times \left(\frac{\overline{A}_x}{A_x}\right) \tag{11}$$

(ii) The interest rate intensity δ depending on i is estimated as follows:

$$\log_e\left(1+i\right) = \frac{-\left\{\dfrac{1}{2}\left(\dfrac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\dfrac{1}{2}m+\dfrac{1}{2}\right\} \pm \sqrt{\left\{\dfrac{1}{2}\left(\dfrac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\dfrac{1}{2}m+\dfrac{1}{2}\right\}^2 + 4\left[\dfrac{1}{12}m^2-\dfrac{1}{12}\left(\dfrac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\dfrac{1}{12}\right]\left(\dfrac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)}}{2\left[\dfrac{1}{12}m^2-\dfrac{1}{12}\left(\dfrac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\dfrac{1}{12}\right]} \tag{12}$$

Proof (i)

The linear interpolation assumption for mortality function is defined as follows:

$$l_{x+s} = (1-s)l_x + sl_{x+1} = l_x - sd_x \tag{13}$$

$$d_x = l_x - l_{x+1} \tag{14}$$

Dividing (13) through by l_x yields

$$\frac{l_{x+s}}{l_x} = (1-s) + s\frac{l_{x+1}}{l_x} = (1-s) + s\left({}_1p_x\right) \tag{15}$$

$$\Rightarrow \left({}_sp_x\right) = 1-s+s\left({}_1p_x\right) = 1-s\left(1-\left({}_1p_x\right)\right) \tag{16}$$

$$\Rightarrow \left({}_sp_x\right) = 1-s\left({}_1q_x\right) \tag{17}$$

Now differentiating $l_{x+s} = l_x - sd_x$ with respect to s

$$\frac{dl_{x+s}}{ds} = -d_x \tag{18}$$

$$\mu_{x+s} = \frac{-1}{l_{x+s}}\frac{dl_{x+s}}{ds} = \frac{-1}{l_{x+s}}\left(-d_x\right) = \frac{d_x}{l_{x+s}} \tag{19}$$

$$\frac{d_x}{l_{x+s}} = \frac{d_x}{l_x}\times\frac{l_x}{l_{x+s}} = \frac{d_x}{l_x}\times\left(\frac{l_{x+s}}{l_x}\right)^{-1} = \left(q_x\right)\times\frac{1}{\left({}_sp_x\right)} = \left(q_x\right)\times\frac{1}{\left\{1-s\left({}_1q_x\right)\right\}} \tag{20}$$

$$f_{T(x)}(s) = \mu_{x+s}\left({}_sp_x\right) \tag{21}$$

$$f_{T(x)}(s) = \left(q_x\right)\times\frac{1}{\left\{1-s\left({}_1q_x\right)\right\}}\left\{1-s\left(1-\left({}_1p_x\right)\right)\right\} = q_x \tag{22}$$

Following Souza (2019), the continuous whole life insurance is defined as

$$\overline{A}x = \int_0^\infty e^{-\delta\xi} f_{T(x)}(\xi)\,d\xi = \int_0^\infty e^{-\delta\xi}\left({}_\xi p_x\right)\mu_{x+\xi}\,d\xi \tag{23}$$

This equation can be approximated as follows

$$\overline{A}x = \sum_{\alpha=0}^{\infty}\left(\int_\alpha^{\alpha+1} e^{-\delta\xi}\frac{l_{x+\xi}}{l_x}\mu_{x+\xi}\,d\xi\right) \tag{24}$$

Let $\xi = \alpha + s$, then $d\xi = ds$ and $s \in \{0,1\}$

Then

$$\bar{A}_x = \sum_{\alpha=0}^{\infty}\left(\int_0^1 e^{-\delta(\alpha+s)}\frac{l_{x+\alpha+s}}{l_x}\mu_{x+\alpha+s}ds\right) = \sum_{\alpha=0}^{\infty}\left(\int_0^1 e^{-\delta(\alpha+s)}\left({}_{\alpha+s}p_x\right)\mu_{x+\alpha+s}ds\right) \quad (25)$$

$$\Rightarrow \bar{A}_x = \sum_{\alpha=0}^{\infty}\left(\int_0^1 e^{-\delta(\alpha+s)}\left({}_{\alpha}p_x\right)\left({}_{s}p_{x+\alpha}\right)\mu_{x+\alpha+s}ds\right) \quad (26)$$

$$\Rightarrow \bar{A}_x = \sum_{\alpha=0}^{\infty}\left({}_{\alpha}p_x\right)\int_0^1 e^{-\delta(\alpha+s)}\left({}_{s}p_{x+\alpha}\right)\mu_{x+\alpha+s}ds \quad (27)$$

We already know that

$$q_x = \mu_{x+s}\left({}_{s}p_x\right) \Rightarrow q_{x+\alpha} = \mu_{x+s+\alpha}\left({}_{s}p_{x+\alpha}\right) \quad (28)$$

Note that

$$e^{-\delta(\alpha+s)} = e^{-\delta(\alpha+1)}e^{\delta(1-s)} = e^{-\delta\alpha-\delta}e^{\delta-\delta s} = e^{-\delta\alpha-\delta s} \quad (29)$$

$$\bar{A}_x = \sum_{\alpha=0}^{\infty}\left({}_{\alpha}p_x\right)\int_0^1 e^{-\delta(\alpha+1)}e^{\delta(1-s)}q_{x+\alpha}ds \quad (30)$$

$$\Rightarrow \bar{A}_x = \left\{\sum_{\alpha=0}^{\infty}\left({}_{\alpha}p_x\right)\left(q_{x+\alpha}\right)e^{-\delta(\alpha+1)}\right\}\times\int_0^1 e^{\delta(1-s)}ds \quad (31)$$

The discrete life insurance is given by

$$A_x = \sum_{\alpha=0}^{\infty}\left({}_{\alpha}p_x\right)\left(q_{x+\alpha}\right)e^{-\delta(\alpha+1)} \quad (32)$$

and

$$\int_0^1 e^{\delta(1-s)}ds = \left[\frac{e^{\delta(1-s)}}{-\delta}\right]_{s=0}^{s=1} = -\left(\frac{e^{\delta(1-1)}}{\delta} - \frac{e^{\delta(1-0)}}{\delta}\right) = -\left(\frac{1-e^{\delta}}{\delta}\right) = \left(\frac{e^{\delta}-1}{\delta}\right) \quad (33)$$

Consequently,

$$\bar{A}_x = A_x\left(\frac{e^{\delta}-1}{\delta}\right) \quad (34)$$

$$\frac{\bar{A}_x}{A_x} = \frac{e^{\delta}-1}{\delta} \Rightarrow \frac{A_x}{\bar{A}_x} = \frac{\delta}{e^{\delta}-1} \tag{35}$$

By definition the discrete discount function $v = \dfrac{1}{1+i}$

The immediate ordinary annuity is defined as

$$a_{\overline{m}|} = \sum_{k=1}^{m} v^k = \left(\frac{1-v^m}{i}\right) = \frac{1}{i}\left(1-\left(\frac{1}{1+i}\right)^m\right) \tag{36}$$

The fully continuous ordinary annuity certain (bank type) is given as

$$\begin{aligned}\bar{a}_{\overline{m}|} &= \int_0^m (1+i)^{-s}\, ds = \int_0^m e^{-\delta s} ds = -\left[\frac{e^{-\delta s}}{\delta}\right]_0^m = -\left[\frac{e^{-\delta m}}{\delta} - \frac{1}{\delta}\right] \\ &= \frac{1-e^{-\delta m}}{\delta} = \frac{1-v^m}{\delta}\end{aligned} \tag{37}$$

Observe that by definition

$$\log_e(1+i) = \delta \Rightarrow e^{\delta} = 1+i \Rightarrow e^{\delta}-1 = i \tag{38}$$

By definition the discrete ordinary annuity is

$$\left(1-v^m\right) = i a_{\overline{m}|} \tag{39}$$

$$\Rightarrow \left(1-v^m\right) = \delta \times \bar{a}_{\overline{m}|} \tag{40}$$

hence

$$\delta \times \bar{a}_{\overline{m}|} = i \times a_{\overline{m}|} \tag{41}$$

$$\frac{m}{a_{\overline{m}|}} = \frac{m \times i}{\left(a_{\overline{m}|}\right) \times i} = \frac{m\left(e^{\delta}-1\right)}{\left(\bar{a}_{\overline{m}|}\right) \times \delta} = \frac{m}{\left(\bar{a}_{\overline{m}|}\right)} \times \frac{\left(e^{\delta}-1\right)}{\delta} \tag{42}$$

Therefore

$$\frac{m}{a_{\overline{m}|}} = \frac{m}{\bar{a}_{\overline{m}|}} \times \left\{\frac{e^{\delta}-1}{\delta}\right\} = \frac{m}{\bar{a}_{\overline{m}|}} \times \frac{\bar{A}_x}{A_x} \tag{43}$$

$$\frac{m}{a_{\overline{m|}}}=\frac{m}{\bar{a}_{\overline{m|}}}\times\left\{\frac{e^{\delta}-1}{\delta}\right\}=\frac{m}{\left\{\frac{1-e^{-m\delta}}{\delta}\right\}}\times\left\{\frac{e^{\delta}-1}{\delta}\right\}=\frac{m\delta}{\left(1-e^{-m\delta}\right)}\times\left\{\frac{e^{\delta}-1}{\delta}\right\}$$
$$=\frac{-m\delta}{\left(e^{-m\delta}-1\right)}\times\frac{\bar{A}_x}{A_x} \tag{44}$$

Proof of (ii)

We use the result in part **(i)** of the theorem to estimate the interest rate intensity

$$\frac{s}{\left[e^{s}-1\right]}=\sum_{m=0}^{\infty}\frac{B_m s^m}{m!} \tag{45}$$

where

$$B_m=\left[\frac{d^m}{d^m}\left(\frac{s}{e^{s}-1}\right)\right]_{s=0} \;;\quad |s|<2\pi \tag{46}$$

$$B_1=\left[\frac{d}{d}\left(\frac{s}{e^{s}-1}\right)\right]_{s=0}=\left[\frac{\left(e^{s}-1\right)-s\left(e^{s}\right)}{\left(e^{s}-1\right)^2}\right]_{s=0}=\frac{1-1-s\left(e^{s}\right)}{\left(e^{s}-1\right)^2}=\frac{0}{0} \tag{47}$$

Now apply L'Hospital rule,

$$B_1=\left[\frac{\left(e^{s}-1\right)-s\left(e^{s}\right)}{\left(e^{s}-1\right)^2}\right]_{s=0}=\left[\frac{e^{s}-\left(se^{s}+e^{s}\right)}{2\left(e^{s}-1\right)}\right]_{s=0}=\left[\frac{-se^{s}}{2\left(e^{s}-1\right)}\right]_{s=0}=\frac{0}{0} \tag{48}$$

again apply L'Hopital's rule.

$$B_1=\left[\frac{\left(e^{s}-1\right)-s\left(e^{s}\right)}{\left(e^{s}-1\right)^2}\right]_{s=0}=\left[\frac{-se^{s}}{2\left(e^{s}-1\right)}\right]_{s=0}=\left[\frac{-se^{s}-e^{s}}{2e^{s}}\right]_{s=0}=\frac{-1}{2} \tag{49}$$

Expanding fully, for $s\neq 0$ yields

$$\frac{s}{\left[e^{s}-1\right]}=\frac{s}{\left(1+s+\frac{s^2}{2!}+\frac{s^3}{3!}+\frac{s^4}{4!}+\ldots-1\right)}=\frac{1}{\left(1+\frac{s}{2!}+\frac{s^2}{3!}+\frac{s^3}{4!}+\ldots\right)} \tag{50}$$

$$\frac{s}{\left[e^{s}-1\right]}=1-\left(\frac{s}{2!}+\frac{s^{2}}{3!}+\frac{s^{3}}{4!}+\ldots\right)+\left(\frac{s}{2!}+\frac{s^{2}}{3!}+\frac{s^{3}}{4!}+\ldots\right)^{2}-\left(\frac{s}{2!}+\frac{s^{2}}{3!}+\frac{s^{3}}{4!}+\ldots\right)^{3}+\ldots$$
$$;s\neq 0 \tag{51}$$

$$\Rightarrow \frac{s}{\left[e^{s}-1\right]}$$
$$=\frac{1}{0!}s^{0}-\frac{1}{2}s+\frac{1}{12}s^{2}-\frac{1}{720}s^{4}+\frac{1}{30240}s^{6}-\frac{1}{1209600}s^{8}+\frac{1}{47900160}s^{10}+\ldots \tag{52}$$

Let $s=-m\delta$ in (52)

$$\left\{\frac{-m\delta}{e^{-m\delta}-1}\right\}=1-\frac{1}{2}(-m\delta)+\frac{1}{12}(-m\delta)^{2}-\frac{1}{720}(-m\delta)^{4}+\frac{1}{30240}(-m\delta)^{6}-\frac{1}{1209600}(-m\delta)^{8}$$
$$+\frac{1}{47900160}(-m\delta)^{10}+\ldots \tag{53}$$

$$\left\{\frac{-m\delta}{e^{-m\delta}-1}\right\}=1+\frac{1}{2}m\delta+\frac{1}{12}m^{2}\delta^{2}-\frac{1}{720}m^{4}\delta^{4}+\frac{1}{30240}m^{6}\delta^{6}-\frac{1}{1209600}m^{8}\delta^{8}$$
$$+\frac{1}{47900160}m^{10}\delta^{10}+\ldots \tag{54}$$

Again let $s=\delta$ in (52)

$$\frac{\delta}{e^{\delta}-1}$$
$$=1-\frac{1}{2}\delta+\frac{1}{12}\delta^{2}-\frac{1}{720}\delta^{4}+\frac{1}{30240}\delta^{6}+\frac{1}{1209600}\delta^{8}+\frac{1}{47900160}\delta^{10}+\ldots \tag{55}$$

$$\Rightarrow \frac{1}{\left\{\frac{\delta}{e^{\delta}-1}\right\}} \times \left\{\frac{-m\delta}{e^{-m\delta}-1}\right\} = \frac{\left\{\begin{array}{l} 1+\frac{1}{2}m\delta+\frac{1}{12}m^2\delta^2-\frac{1}{720}m^4\delta^4+\frac{1}{30240}m^6\delta^6-\frac{1}{1209600}m^8\delta^8 \\ +\frac{1}{47900160}m^{10}\delta^{10}+\ldots \end{array}\right\}}{\left\{1-\frac{1}{2}\delta+\frac{1}{12}\delta^2-\frac{1}{720}\delta^4+\frac{1}{30240}\delta^6+\frac{1}{1209600}\delta^8-\frac{1}{47900160}\delta^{10}+\ldots\right\}} \quad (56)$$

Using the quadratic estimation, we can ignore all the terms from the fourth term and above in the numerator and in the denominator to have

$$\frac{m}{a_{\overline{m}|}} = \frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^2\delta^2}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^2} \quad (57)$$

Subtracting 1 from both sides of (57), yields

$$\frac{m}{a_{\overline{m}|}}-1 = \frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^2\delta^2}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^2}-1 \quad (58)$$

$$\Rightarrow \frac{m}{a_{\overline{m}|}}-1 = \frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^2\delta^2-\left(1-\frac{1}{2}\delta+\frac{1}{12}\delta^2\right)}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^2} \quad (59)$$

$$\Rightarrow \frac{m}{a_{\overline{m}|}}-1 = \frac{1+\frac{1}{2}m\delta+\frac{1}{12}m^2\delta^2-1+\frac{1}{2}\delta-\frac{1}{12}\delta^2}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^2} \quad (60)$$

$$\Rightarrow \frac{m}{a_{\overline{m}|}}-1 = \frac{\left(\frac{1}{2}m+\frac{1}{2}\right)\delta+\left(\frac{1}{12}m^2-\frac{1}{12}\right)\delta^2}{1-\frac{1}{2}\delta+\frac{1}{12}\delta^2} \quad (61)$$

Let $\theta = \dfrac{m}{a_{\overline{m}|}} - 1$; $\theta = \dfrac{m - a_{\overline{m}|}}{a_{\overline{m}|}}$

$$\Rightarrow \theta = \frac{\left(\frac{1}{2}m + \frac{1}{2}\right)\delta + \left(\frac{1}{12}m^2 - \frac{1}{12}\right)\delta^2}{1 - \frac{1}{2}\delta + \frac{1}{12}\delta^2} \tag{62}$$

$$\Rightarrow \theta\left(1 - \frac{1}{2}\delta + \frac{1}{12}\delta^2\right) = \left(\frac{1}{2}m + \frac{1}{2}\right)\delta + \left(\frac{1}{12}m^2 - \frac{1}{12}\right)\delta^2 \tag{63}$$

$$\Rightarrow \theta - \frac{1}{2}\delta\theta + \frac{1}{12}\theta\delta^2 = \frac{1}{2}m\delta + \frac{1}{2}\delta + \frac{1}{12}m^2\delta^2 - \frac{1}{12}\delta^2 \tag{64}$$

$$\Rightarrow \left(\frac{1}{12}m^2 - \frac{1}{12}\theta - \frac{1}{12}\right)\delta^2 + \left(\frac{1}{2}\theta + \frac{1}{2}m + \frac{1}{2}\right)\delta - \theta = 0 \tag{65}$$

$$\Rightarrow \delta = \frac{-\left(\frac{1}{2}\theta + \frac{1}{2}m + \frac{1}{2}\right) \pm \sqrt{\left(\frac{1}{2}\theta + \frac{1}{2}m + \frac{1}{2}\right)^2 + 4\left(\frac{1}{12}m^2 - \frac{1}{12}\theta - \frac{1}{12}\right)\theta}}{2\left(\frac{1}{12}m^2 - \frac{1}{12}\theta - \frac{1}{12}\right)} \tag{66}$$

$$\Rightarrow \delta_{est} = \frac{-\left\{\frac{1}{2}\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) + \frac{1}{2}m + \frac{1}{2}\right\} \pm \sqrt{\left\{\frac{1}{2}\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) + \frac{1}{2}m + \frac{1}{2}\right\}^2 + 4\left[\frac{1}{12}m^2 - \frac{1}{12}\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) - \frac{1}{12}\right]\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right)}}{2\left[\frac{1}{12}m^2 - \frac{1}{12}\left(\frac{m - a_{\overline{m}|}}{a_{\overline{m}|}}\right) - \frac{1}{12}\right]} \tag{67}$$

Since $\delta > 0$, it follows that

$$\Rightarrow \delta_{est} = \frac{-\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)+\frac{1}{2}m+\frac{1}{2}\right\} + \sqrt{\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}^2 + 4\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)-\frac{1}{12}\right]\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)}}{2\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)-\frac{1}{12}\right]} \quad (68)$$

Furthermore, to justify why the term must possibly start at 1, we must have that

$$\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)-\frac{1}{12} > 0 \quad (69)$$

$$\frac{1}{12}m^2 - m\frac{1}{12a_{\overline{m|}}} + \left(a_{\overline{m|}}\frac{1}{12a_{\overline{m|}}} - \frac{1}{12}\right) > 0 \quad (70)$$

$$m^2 - m\frac{1}{a_{\overline{m|}}} > 0 \quad (71)$$

$$\Rightarrow m\left(m - \frac{1}{a_{\overline{m|}}}\right) > 0 \quad (72)$$

The zeros of the corresponding quadratic equation are

$m = 0$ or $m = \frac{1}{a_{\overline{m|}}}$;

The solution to the quadratic inequality

$m < 0$ or $m > \frac{1}{a_{\overline{m|}}}$ and this accounts for why m must possibly start at 1

By definition

$$\delta_{EXACT} = \ln(1+i) \tag{73}$$

Therefore, the estimated discount function is expressible in the form

$$v_{est} = \exp(-)\left\{ \frac{-\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\} \pm \sqrt{\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}^2 + 4\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right]\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)}}{2\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right]} \right\} \tag{74}$$

This completes the proof

As the term m increases, the immediate annuity $a_{\overline{m}|}$ also increases. The increase is justified as follows:

$$a_{\overline{m}|} = v\times 1 + v^2\times 1 + v^3\times 1 + \ldots + v^{m-1}\times 1 + v^m\times 1 \tag{75}$$

$$\Rightarrow a_{\overline{m}|} = v\left(1+v^1+v^2+\ldots+v^{m-2}+v^{m-1}\right) \tag{76}$$

$$\Rightarrow a_{\overline{m}|} = v\left(\frac{1-v^m}{1-v}\right) = \frac{1}{1+i}\left(\frac{1-v^m}{d}\right) \tag{77}$$

$$\Rightarrow a_{\overline{m}|} = \frac{1}{1+i}\left(\frac{1-v^m}{\frac{i}{1+i}}\right) = \frac{1-v^m}{i} \tag{78}$$

Define

$$i^{(m)} = \lim_{m\to\infty} m \times \left(\frac{A\left(t+\frac{1}{m}\right) - A(t)}{A(t)} \right) \tag{79}$$

and let $\Delta = \frac{1}{m}$. Then as $\Delta \to 0$ and $m \to \infty$

$$i^{(m)} = \lim_{\Delta\to 0} \frac{1}{\Delta} \times \left(\frac{A(t+\Delta) - A(t)}{A(t)} \right) \tag{80}$$

$$i^{(m)} = \lim_{\Delta\to 0} \frac{1}{A(t)} \times \left(\frac{A(t+\Delta) - A(t)}{\Delta} \right) \tag{81}$$

$$i^{(m)} = \frac{1}{A(t)} \times \lim_{\Delta\to 0} \left(\frac{A(t+\Delta) - A(t)}{\Delta} \right) \tag{82}$$

$$i^{(m)} = \frac{A'(t)}{A(t)} = \delta(t) \tag{83}$$

Data analysis and presentation

The table below shows the computation of the estimated interest rate intensities in comparison to the exact value

Table 1: Estimated force of interest and corresponding present value function

m	$a_{\overline{m}\|}$	δ_{est}	v_{est}	$\lvert\delta_{exaxt} - \delta_{est}\rvert$
1	0.9523809524	0.0487901646	0.9523809520	-0.0000000004
2	1.8594104308	0.0487900890	0.9523810240	0.0000000752
3	2.7232480294	0.0487898670	0.9523812354	0.0000002972
4	3.5459505042	0.0487894177	0.9523816633	0.0000007465
5	4.3294766706	0.0487886654	0.9523823798	0.0000014988
6	5.0756920673	0.0487875391	0.9523834524	0.0000026251
7	5.7863733974	0.0487859725	0.9523849444	0.0000041917
8	6.4632127594	0.0487839038	0.9523869147	0.0000062604
9	7.1078216756	0.0487812750	0.9523894183	0.0000088892
10	7.7217349292	0.0487780325	0.9523925065	0.0000121317

11	8.3064142183	0.0487741259	0.9523962271	0.0000160383
12	8.8632516364	0.0487695088	0.9524006244	0.0000206554
13	9.3935729871	0.0487641380	0.9524057396	0.0000260262
14	9.8986409401	0.0487579735	0.9524116107	0.0000321907
15	10.3796580382	0.0487509785	0.9524182728	0.0000391857
16	10.8377695602	0.0487431190	0.9524257584	0.0000470452
17	11.2740662478	0.0487343640	0.9524340969	0.0000558002
18	11.6895869027	0.0487246851	0.9524433154	0.0000654791
19	12.0853208597	0.0487140565	0.9524534387	0.0000761077
20	12.4622103425	0.0487024547	0.9524644889	0.0000877095
21	12.8211527072	0.0486898589	0.9524764860	0.0001003053
22	13.1630025783	0.0486762503	0.9524894480	0.0001139139
23	13.4885738841	0.0486616123	0.9525033906	0.0001285519
24	13.7986417943	0.0486459304	0.9525183278	0.0001442338
25	14.0939445660	0.0486291922	0.9525342713	0.0001609720
26	14.3751853010	0.0486113870	0.9525512315	0.0001787772
27	14.6430336200	0.0485925061	0.9525692168	0.0001976581
28	14.8981272571	0.0485725423	0.9525882338	0.0002176219
29	15.1410735782	0.0485514904	0.9526082879	0.0002386738
30	15.3724510269	0.0485293463	0.9526293827	0.0002608179
31	15.5928105018	0.0485061080	0.9526515205	0.0002840562
32	15.8026766684	0.0484817744	0.9526747022	0.0003083898
33	16.0025492080	0.0484563462	0.9526989274	0.0003338180
34	16.1929040076	0.0484298250	0.9527241944	0.0003603392
35	16.3741942929	0.0484022141	0.9527505003	0.0003879501
36	16.5468517076	0.0483735176	0.9527778413	0.0004166466
37	16.7112873405	0.0483437409	0.9528062123	0.0004464233
38	16.8678927053	0.0483128905	0.9528356073	0.0004772737
39	17.0170406717	0.0482809737	0.9528660192	0.0005091905
40	17.1590863540	0.0482479990	0.9528974402	0.0005421652
41	17.2943679562	0.0482139756	0.9529298615	0.0005761886
42	17.4232075773	0.0481789137	0.9529632736	0.0006112505
43	17.5459119784	0.0481428242	0.9529976662	0.0006473400
44	17.6627733128	0.0481057188	0.9530330282	0.0006844454
45	17.7740698217	0.0480676098	0.9530693481	0.0007225544
46	17.8800664968	0.0480285103	0.9531066134	0.0007616539
47	17.9810157113	0.0479884338	0.9531448113	0.0008017304

48	18.0771578203	0.0479473945	0.9531839285	0.0008427697
49	18.1687217336	0.0479054072	0.9532239510	0.0008847570
50	18.2559254606	0.0478624870	0.9532648644	0.0009276772
51	18.3389766291	0.0478186495	0.9533066541	0.0009715147
52	18.4180729801	0.0477739107	0.9533493048	0.0010162535
53	18.4934028382	0.0477282870	0.9533928011	0.0010618772
54	18.5651455602	0.0476817951	0.9534371271	0.0011083691
55	18.6334719621	0.0476344521	0.9534822669	0.0011557121
56	18.6985447258	0.0475862750	0.9535282039	0.0012038892
57	18.7605187865	0.0475372815	0.9535749217	0.0012528827
58	18.8195417014	0.0474874893	0.9536224036	0.0013026749
59	18.8757540013	0.0474369160	0.9536706326	0.0013532482
60	18.9292895251	0.0473855798	0.9537195917	0.0014045844
61	18.9802757382	0.0473334988	0.9537692637	0.0014566654
62	19.0288340363	0.0472806910	0.9538196314	0.0015094732
63	19.0750800346	0.0472271748	0.9538706776	0.0015629894
64	19.1191238425	0.0471729684	0.9539223848	0.0016171958
65	19.1610703262	0.0471180902	0.9539747358	0.0016720740
66	19.2010193583	0.0470625584	0.9540277132	0.0017276058
67	19.2390660555	0.0470063913	0.9540812997	0.0017837729
68	19.2753010052	0.0469496072	0.9541354780	0.0018405570
69	19.3098104812	0.0468922240	0.9541902308	0.0018979402
70	19.3426766487	0.0468342601	0.9542455411	0.0019559041
71	19.3739777607	0.0467757333	0.9543013917	0.0020144309
72	19.4037883435	0.0467166614	0.9543577657	0.0020735028
73	19.4321793748	0.0466570623	0.9544146462	0.0021331019
74	19.4592184522	0.0465969535	0.9544720167	0.0021932107
75	19.4849699545	0.0465363525	0.9545298604	0.0022538117
76	19.5094951947	0.0464752766	0.9545881610	0.0023148876
77	19.5328525664	0.0464137428	0.9546469022	0.0023764214
78	19.5550976823	0.0463517681	0.9547060680	0.0024383961
79	19.5762835069	0.0462893692	0.9547656425	0.0025007950
80	19.5964604828	0.0462265626	0.9548256099	0.0025636016
81	19.6156766503	0.0461633646	0.9548859549	0.0026267996
82	19.6339777622	0.0460997913	0.9549466621	0.0026903729
83	19.6514073925	0.0460358586	0.9550077164	0.0027543056
84	19.6680070405	0.0459715820	0.9550691029	0.0028185822

85	19.6838162291	0.0459069771	0.9551308071	0.0028831871
86	19.6988725991	0.0458420588	0.9551928146	0.0029481054
87	19.7132119992	0.0457768422	0.9552551111	0.0030133220
88	19.7268685706	0.0457113418	0.9553176827	0.0030788224
89	19.7398748292	0.0456455720	0.9553805158	0.0031445922
90	19.7522617421	0.0455795471	0.9554435968	0.0032106171
91	19.7640588020	0.0455132808	0.9555069126	0.0032768834
92	19.7752940971	0.0454467867	0.9555704503	0.0033433775
93	19.7859943782	0.0453800783	0.9556341970	0.0034100859
94	19.7961851221	0.0453131685	0.9556981404	0.0034769957
95	19.8058905925	0.0452460703	0.9557622682	0.0035440939
96	19.8151338976	0.0451787962	0.9558265684	0.0036113680
97	19.8239370453	0.0451113584	0.9558910294	0.0036788058
98	19.8323209955	0.0450437690	0.9559556397	0.0037463952
99	19.8403057100	0.0449760398	0.9560203880	0.0038141244
100	19.8479102000	0.0449081822	0.9560852635	0.0038819820

$\delta_{exact} = 0.0487901642$ and $v_{exact} = 0.9523809524$ at $i = 0.05$

Table below shows the computed results for the interest and discount rates compounded many times

Table 2: Compounded interest and discount rates

m	$i^{(m)}$	$d^{(m)}$
1	0.0500000004	0.0476190480
2	0.0493900761	0.0481997807
3	0.0491887684	0.0483952672
4	0.0490881821	0.0484930729
5	0.0490274749	0.0485514044
6	0.0489864298	0.0485897236
7	0.0489563733	0.0486163617
8	0.0489329484	0.0486354638
9	0.0489137149	0.0486493129
10	0.0488971909	0.0486592608
11	0.0488824184	0.0486661530
12	0.0488687459	0.0486705402
13	0.0488557117	0.0486727929
14	0.0488429771	0.0486731670
15	0.0488302863	0.0486718423

16	0.0488174411	0.0486689477
17	0.0488042849	0.0486645767
18	0.0487906917	0.0486587975
19	0.0487765588	0.0486516609
20	0.0487618011	0.0486432046
21	0.0487463478	0.0486334572
22	0.0487301395	0.0486224405
23	0.0487131258	0.0486101713
24	0.0486952643	0.0485966632
25	0.0486765189	0.0485819269
26	0.0486568590	0.0485659718
27	0.0486362589	0.0485488058
28	0.0486146969	0.0485304365
29	0.0485921552	0.0485108708
30	0.0485686191	0.0484901159
31	0.0485440768	0.0484681787
32	0.0485185193	0.0484450667
33	0.0484919396	0.0484207875
34	0.0484643333	0.0483953495
35	0.0484356977	0.0483687613
36	0.0484060321	0.0483410322
37	0.0483753373	0.0483121720
38	0.0483436158	0.0482821912
39	0.0483108713	0.0482511007
40	0.0482771090	0.0482189123
41	0.0482423353	0.0481856381
42	0.0482065577	0.0481512909
43	0.0481697847	0.0481158839
44	0.0481320257	0.0480794311
45	0.0480932911	0.0480419468
46	0.0480535922	0.0480034457
47	0.0480129409	0.0479639433
48	0.0479713499	0.0479234551
49	0.0479288324	0.0478819972
50	0.0478854024	0.0478395861
51	0.0478410743	0.0477962386
52	0.0477958630	0.0477519718
53	0.0477497839	0.0477068030

54	0.0477028528	0.0476607499
55	0.0476550857	0.0476138304
56	0.0476064991	0.0475660624
57	0.0475571098	0.0475174643
58	0.0475069348	0.0474680544
59	0.0474559911	0.0474178511
60	0.0474042964	0.0473668731
61	0.0473518679	0.0473151391
62	0.0472987235	0.0472626676
63	0.0472448809	0.0472094776
64	0.0471903578	0.0471555876
65	0.0471351722	0.0471010165
66	0.0470793418	0.0470457830
67	0.0470228848	0.0469899056
68	0.0469658187	0.0469334031
69	0.0469081616	0.0468762937
70	0.0468499311	0.0468185961
71	0.0467911449	0.0467603284
72	0.0467318206	0.0467015088
73	0.0466719756	0.0466421553
74	0.0466116274	0.0465822858
75	0.0465507931	0.0465219180
76	0.0464894897	0.0464610693
77	0.0464277341	0.0463997571
78	0.0463655432	0.0463379985
79	0.0463029332	0.0462758104
80	0.0462399207	0.0462132095
81	0.0461765217	0.0461502124
82	0.0461127522	0.0460868352
83	0.0460486278	0.0460230941
84	0.0459841640	0.0459590047
85	0.0459193761	0.0458945825
86	0.0458542790	0.0458298430
87	0.0457888875	0.0457648011
88	0.0457232162	0.0456994715
89	0.0456572792	0.0456338689
90	0.0455910907	0.0455680074
91	0.0455246643	0.0455019010

92	0.0454580136	0.0454355635
93	0.0453911519	0.0453690083
94	0.0453240920	0.0453022486
95	0.0452568468	0.0452352972
96	0.0451894287	0.0451681670
97	0.0451218499	0.0451008701
98	0.0450541223	0.0450334188
99	0.0449862577	0.0449658249
100	0.0449182674	0.0448981000

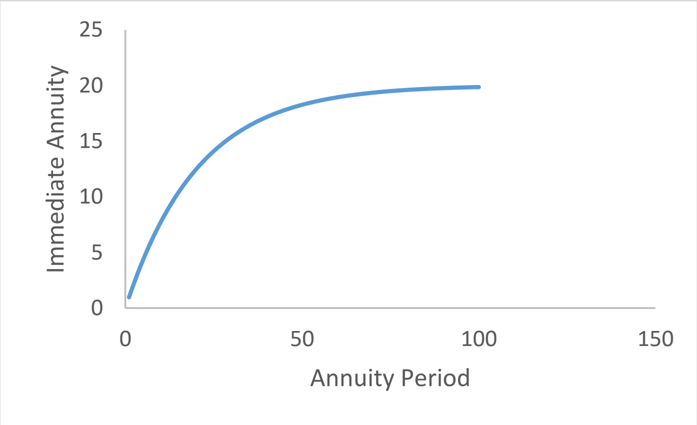

Figure 1: The graph of immediate annuity and annuity period.

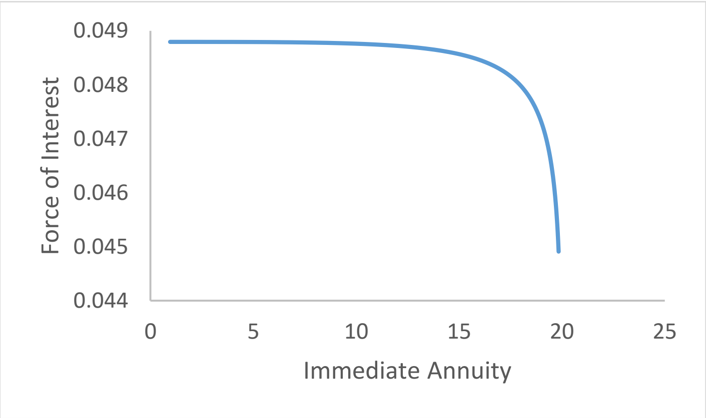

Figure 2: The graph of force of interest and immediate annuity.

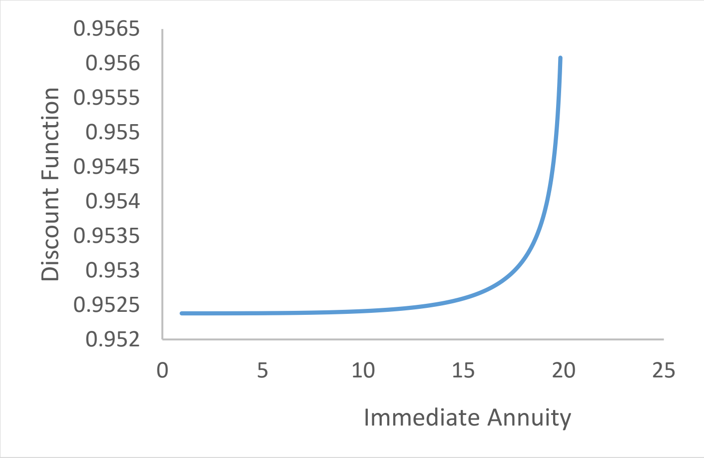

Figure 3: The graph of discount function and immediate annuity

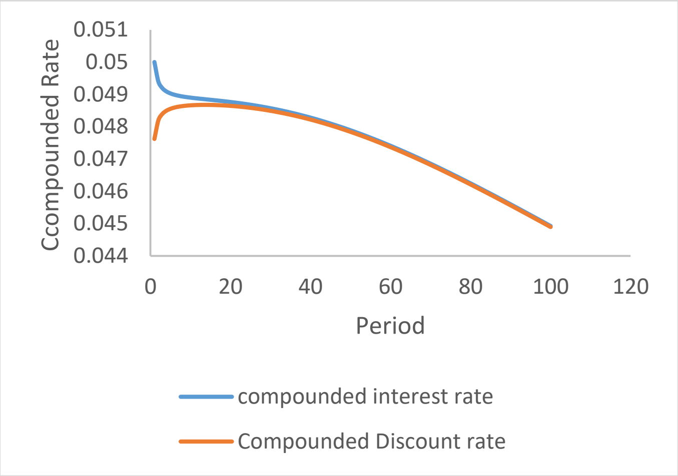

Figure 4: The graph of discount function and immediate annuity

Discussion of Results

Following the definition of the power series, the Bernoulli equation (84) can be expressed for the purpose of interest rate computations as follows.

$$\frac{\delta}{e^{\delta}-1}=\sum_{m=0}^{\infty}\frac{B_m\delta^m}{m!} \tag{84}$$

$$\delta=\left(e^{\delta}-1\right)\sum_{m=0}^{\infty}\frac{B_m\delta^m}{m!} \tag{85}$$

$$\Rightarrow \delta = \left(\delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \frac{\delta^4}{4!} + \ldots \right) \sum_{m=0}^{\infty} \frac{B_m \delta^m}{m!} \tag{86}$$

$$\Rightarrow \delta = \sum_{r=1}^{\infty} \frac{\delta^r}{r!} \sum_{m=0}^{\infty} \frac{B_m \delta^m}{m!} \tag{87}$$

$$\Rightarrow \delta = \sum_{r=0}^{\infty} \frac{\delta^{r+1}}{(r+1)!} \sum_{m=0}^{\infty} \frac{B_m \delta^m}{m!} \tag{88}$$

Therefore if i is the interest rate, then by definition,

$$\log_e(1+i) = \sum_{r=0}^{\infty} \frac{\delta^{r+1}}{(r+1)!} \sum_{m=0}^{\infty} \frac{B_m \delta^m}{m!} \tag{89}$$

The Cauchy product of two infinite series $\left(\sum_{K=0}^{\infty} A_K \right)$ and $\left(\sum_{M=0}^{\infty} B_M \right)$ is given by

$$\sum_{N=0}^{\infty} C_N = \left(\sum_{K=0}^{\infty} A_K \right) \left(\sum_{M=0}^{\infty} B_M \right) \tag{90}$$

$$C_N = A_0 B_N + A_1 B_{N-1} + \ldots + A_N B_0 = \sum_{K=0}^{N} A_K B_{N-K} \tag{91}$$

Therefore, based on the Cauchy product, the force of interest is given by

$$\delta = \sum_{N=0}^{\infty} \sum_{K=0}^{N} \frac{\delta^{N+1-K}}{(N+1-K)!} \frac{B_K \delta^K}{K!} \tag{92}$$

$$\Rightarrow \delta = \sum_{N=0}^{\infty} \sum_{K=0}^{N} \frac{B_K \delta^{N+1}}{K! \times (N+1-K)!} \tag{93}$$

$$\Rightarrow \delta = \sum_{N=0}^{\infty} \sum_{K=0}^{N} \frac{(N+1)! B_K}{K! \times (N+1-K)!} \frac{\delta^{N+1}}{(N+1)!} \tag{94}$$

$$\Rightarrow \delta = \sum_{N=0}^{\infty} \sum_{K=0}^{N} \binom{N+1}{K} B_K \frac{\delta^{N+1}}{(N+1)!} \tag{95}$$

$$\Rightarrow \sum_{N=0}^{\infty}\sum_{K=0}^{N}\binom{N+1}{K}B_K\frac{\delta^{N+1}}{(N+1)!}$$

$$\cong \left\{ \frac{-\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}+\sqrt{\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}^2+4\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right]\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)}}{2\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right]} \right\} \tag{96}$$

From the same Table 1, the value of the estimated force of interest steadily reduces at different levels of increasing immediate annuity $a_{\overline{m}|}$. This is because, as the term $m \to \infty$, $v^m \to 0$. From our observation, $\left|\delta_{est} - \delta_{exact}\right| < \varepsilon$ where $\varepsilon > 0$ is a small number. We specify the tolerance limit a priori $\varepsilon = 0.4\%$ and the absolute value of the difference falls within the tolerance limit. A defined acceptable range of precision was set and in the last column of Table 1, the technique of modelling the interest rate intensity employed has shown a clear extent of departure of the estimated from the exact result. If the estimated rate is beyond the acceptable limits, the estimation is ignored. The analytical justification of having smooth progression of interest rates certainly satisfies the practical need to have smooth rates to ascertain that premium rates computed do not exhibit irregularities.

However, the present value function v_{est} progressively increases as $a_{\overline{m}|}$ increases. The interest rate intensity offers an acceptable method to perceive the rate at which an investment of 1 increases on a continuous basis. The traditional compounding under specified period of time always results in discrete changes to the accumulated amount. However, it is observed in the Table that the continuous compounding leads to a smooth and uninterrupted growth. Equation (96) has two implications on both the interest rate and discount rate compounded m The interest rate compounded m times expressed in terms of the estimated force of interest becomes.

$$i^{(m)} = m\left(\exp\left(\frac{1}{m}\right)\left\{ \frac{ -\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\} \pm \sqrt{ \left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}^{2} +4\left[\frac{1}{12}m^{2}-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right]\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right) } }{ 2\left[\frac{1}{12}m^{2}-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right] } \right\} - 1 \right) \tag{97}$$

Consequently,

$$i^{(m)} = \log_e\left[\lim_{m\to\infty}\left(1+\frac{i^{(m)}}{m}\right)^{m}\right] = \log_e\left[\lim_{m\to\infty}\left(\left(1+\frac{1}{\frac{m}{i^{(m)}}}\right)^{\frac{m}{i^{(m)}}}\right)^{i^{(m)}}\right] \tag{98}$$

while the discount function becomes

$$d^{(m)} = m\left(1-\exp\left(\frac{-1}{m}\right)\left\{\frac{-\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)+\frac{1}{2}m+\frac{1}{2}\right\} \pm \sqrt{\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}^2 +4\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)-\frac{1}{12}\right]\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)}}{2\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m|}}}{a_{\overline{m|}}}\right)-\frac{1}{12}\right]}\right\}\right) \quad (99)$$

In Figure 4, the significant difference of nominal rate changes between $i^{(m)}$ and $d^{(m)}$ was shown to converge at some point. Table 2, shows the trends of the estimated force of interest and the discount function. The estimated values taper to the true value. In Figure 2, the force of interest declines as the immediate annuity increases while in Figure 3 the force of discount increases as annuity increases**.** Taking the limits in equation (97) and (99) results to.

$$\lim_{m\to\infty} m \times \left(\exp\left(\frac{1}{m}\right) \left\{ \frac{ -\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\} \pm \sqrt{ \left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}^2 +4\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right]\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right) } }{ 2\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right] } \right\} -1 \right) \tag{100}$$

$$= \delta_{est}$$

$$\lim_{m\to\infty} m \times \left(1-\exp\left(\frac{-1}{m}\right) \left\{ \frac{ -\left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\} \pm \sqrt{ \left\{\frac{1}{2}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)+\frac{1}{2}m+\frac{1}{2}\right\}^2 +4\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right]\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right) } }{ 2\left[\frac{1}{12}m^2-\frac{1}{12}\left(\frac{m-a_{\overline{m}|}}{a_{\overline{m}|}}\right)-\frac{1}{12}\right] } \right\} \right) \tag{101}$$

$$= \delta_{est}$$

This holds true because since δ is small, then $\frac{\delta}{m}$ is smaller and we are permitted to take the first order linear approximation $e^{\frac{\delta}{m}} = 1 + \frac{\delta}{m}$ is appropriate, hence the result on cross-multiplication by m. This further explains why in Figure 4, the compounded rates $i^{(m)}$ and $d^{(m)}$ converges to the same δ from two opposite directions as the period approaches infinity. Furthermore, the effective annual rate of interest (EA) can be defined as

$$EA = \left(1 + \frac{i}{m}\right)^m - 1 \tag{102}$$

Consequently, the annual bonus interest in excess of interest rate obtained through compounding is expressible in the form

$$BI = \left(1 + \frac{i}{m}\right)^m - 1 - i \tag{103}$$

This expression describes the merits of high frequency of compounding because it improves the effective annual interest rate.

$$EA = \lim_{m \to \infty} \left(1 + \frac{i}{m}\right)^m - 1 = e^i - 1 \tag{104}$$

The maximum bonus interest is achieved under continuous compounding. Consequently, high level of compounding frequency gives further annual bonus interest through the peaked value at high compounding frequency. Suppose the principal α is contributed every year τ under continuous compounding at an annual interest rate i, then the accumulation function can be defined as

$$A(\tau) = \alpha e^{i\tau} + \alpha e^{i(\tau-1)} + \alpha e^{i(\tau-2)} + \alpha e^{i(\tau-3)} + \alpha e^{i(\tau-4)} + \ldots + \alpha \tag{105}$$

$$\Rightarrow A(\tau) = \sum_{m=0}^{\tau} \alpha e^{im} = \frac{\alpha\left(e^{i\tau+1} - 1\right)}{e^i - 1} \tag{106}$$

Since the amount invested at any time is the total principal $\alpha\tau$, the net appreciation n_τ representing the increase in value above the total principal is given by

$$n_\tau = \frac{\alpha\left(e^{i\tau+1} - 1\right)}{e^i - 1} - \alpha\tau \tag{107}$$

Implications of the force of interest for life insurers

The force of interest has several important implications for life insurers, particularly in terms of pricing, reserving, investment strategy and regulatory compliance as follows: The force of interest influences the pricing of life insurance policies by determining how future cash flows (like premiums, benefits, and claims) are discounted to the present. This has several implications: By using the force of interest, insurers can more precisely determine the present value of future policyholder benefits, ensuring that premiums are set at levels that reflect the true cost of providing insurance coverage over time. Since the force of interest affects the discounting of future obligations, inaccurate modeling or assumptions about the force of interest can lead to mispricing. This could result in underpricing (leading to inadequate reserves) or overpricing (leading to loss of competitiveness). For products with long durations (like whole life or universal life), the force of interest helps insurers calculate the time value of future premiums, allowing for more accurate premium setting and better matching of cash flows over the life of the policy.

The force of interest plays a critical role in determining the reserves life insurers must hold to meet future policyholder obligations. Implications here include: accurate reserve setting relies on discounting future liabilities using an appropriate force of interest. If the insurer's assumptions about the force of interest are too low, reserves may be underestimated, which could lead to solvency issues in the future. Regulatory frameworks like Solvency II in Europe or IFRS 17 internationally often requires insurers to apply a market-consistent discount rate when calculating reserves. The force of interest, typically reflecting a risk-free or market-based rate, ensures that reserves align with the real cost of capital, helping to avoid solvency risks. Modeling the force of interest allows insurers to conduct stress testing under different interest rate scenarios. This helps assess the robustness of their reserves in various market conditions, including interest rate shocks or prolonged low-rate environments.

The force of interest influences how insurers approach their investment strategies and asset-liability management (ALM): Insurers need to ensure that their asset portfolio provides returns that align with the force of interest assumptions. If the force of interest reflects a low-rate environment, insurers may need to adjust their investment strategies (by holding more riskier assets or longer-duration bonds) to meet future obligations. Life insurers must match the duration and risk profile of their liabilities with their asset portfolios. The force of interest informs the discounting of liabilities, and thus helps insurers determine how to structure their investments to meet these liabilities at the appropriate time. Life insurers with long-duration liabilities (such as annuities or whole life policies) are sensitive to changes in interest rates. Using the force of interest enables insurers to model and hedge against the risks associated with fluctuating interest rates, which could impact the present value of future liabilities and the performance of assets.

The force of interest reflects continuous compounding of interest rates, and as such, has important implications for how life insurers manage interest rate risk—changes in interest rates (a rise or fall in risk-free rates) —that directly affect the discount rate used for future liabilities. A higher force of interest reduces the present value of future liabilities, potentially improving the insurer's solvency position. Conversely, a lower force of interest increases the present value of liabilities, which may require higher reserves. A sudden change in interest rates can significantly impact the profitability of life insurance products. For instance, if the insurer has not adequately priced for interest rate fluctuations, the changes in the force of interest may affect the balance between the premiums collected and the benefits paid out. Some life insurance products (e.g., variable annuities, universal life policies) have embedded options like guaranteed minimum death benefits or guaranteed interest rates. The force of interest helps insurers price and reserve for these options by modeling how changes in interest rates affect the value of these guarantees.

Regulatory frameworks often mandate that life insurers apply the force of interest in their valuation methodologies. This ensures consistency, transparency, and risk management in financial statements of insurers: Under regulatory standards like Solvency II or IFRS 17, insurers are required to discount future liabilities using a rate that reflects market conditions. The force of interest is typically used to meet these requirements, ensuring that insurers comply with capital adequacy and solvency standards. Life insurers must report their balance sheets, income statements, and cash flow projections in accordance with generally accepted accounting principles (GAAP) or international financial reporting standards (IFRS). The force of interest is used to discount future liabilities, which impacts the insurer's reported financial position, profitability, and risk profile. The force of interest is also important for calculating the embedded value of a life insurer, which is the present value of future profits from in-force policies. A market-consistent force of interest helps ensure that the embedded value reflects current financial conditions, which is important for investors and regulators. Life insurers offering products with long-term guarantees, such as whole life, annuities, or universal life with investment components, are particularly affected by the force of interest: The force of interest affects the valuation of guarantees embedded in life insurance products, such as minimum death benefits, cash value growth, or guaranteed annuity rates. These guarantees are typically sensitive to interest rate assumptions, so the force of interest needs to be carefully modeled to ensure the insurer is holding sufficient reserves. Changes in the force of interest can influence policyholder behavior, such as lapses, surrenders, or premium payments. For example, when interest rates are low, policyholders may be more inclined to surrender policies early if they are dissatisfied with the low returns. Insurers need to account for such behavior in their pricing and reserving models.

Conclusion

Life insurers must manage their capital and risk exposure effectively in response to changes in interest rates and the force of interest: The insurer's capital requirements are impacted by the force of interest, as changes in interest rates affect the present value of liabilities. Insurers may need to adjust their capital buffers or risk management strategies in response to shifts in interest rate assumptions or market conditions. Life insurers with significant interest rate exposure may use derivatives or other financial instruments to hedge against risks arising from changes in the force of interest. Effective hedging strategies can protect the insurer's financial position from adverse movements in interest rates. The force of interest has wide-ranging implications for life insurers, touching on areas such as pricing, reserving, investment strategy, regulatory compliance, and risk management. Accurate modeling of the force of interest is essential for ensuring that insurers can meet their long-term obligations, manage their capital efficiently, and maintain solvency under varying interest rate environments. By incorporating the force of interest into their models, insurers can better align their assets and liabilities, optimize profitability, and comply with regulatory standards.

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